General Certificate of Education
Advanced Level Examination
June 2011

Mathematics MFP3

Unit Further Pure 3

Thursday 16 June 2011 1.30 pm to 3.00 pm

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer the questions in the spaces provided. Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

1 The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$$

where $$\mathrm{f}(x, y) = x + \ln(1 + y)$$

and $$y(2) = 1$$

Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = h\mathrm{f}(x_r, y_r)$ and $k_2 = h\mathrm{f}(x_r + h, y_r + k_1)$ and $h = 0.2$, to obtain an approximation to $y(2.2)$, giving your answer to four decimal places. *(5 marks)*

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2 (a) Find the values of the constants p and q for which $p + qx\mathrm{e}^{-2x}$ is a particular integral of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 4 - 9\mathrm{e}^{-2x}$$ *(5 marks)*

(b) Hence find the general solution of this differential equation. *(3 marks)*

(c) Hence express y in terms of x, given that $y = 4$ when $x = 0$ and that $\frac{\mathrm{d}y}{\mathrm{d}x} \to 0$ as $x \to \infty$. *(4 marks)*

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3 (a) Find $\int x^2 \ln x \, \mathrm{d}x$. *(3 marks)*

(b) Explain why $\int_0^{\mathrm{e}} x^2 \ln x \, \mathrm{d}x$ is an improper integral. *(1 mark)*

(c) Evaluate $\int_0^{\mathrm{e}} x^2 \ln x \, \mathrm{d}x$, showing the limiting process used. *(3 marks)*

4 By using an integrating factor, find the solution of the differential equation

$$\frac{dy}{dx} + (\cot x)y = \sin 2x, \quad 0 < x < \frac{\pi}{2}$$

given that $y = \frac{1}{2}$ when $x = \frac{\pi}{6}$. *(10 marks)*

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5 (a) Given that $y = \ln(1 + 2\tan x)$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(You may leave your expression for $\frac{d^2y}{dx^2}$ unsimplified.) *(4 marks)*

(b) Hence, using Maclaurin's theorem, find the first two non-zero terms in the expansion, in ascending powers of x, of $\ln(1 + 2\tan x)$. *(2 marks)*

(c) Find

$$\lim_{x \to 0} \left[\frac{\ln(1 + 2\tan x)}{\ln(1 - x)}\right]$$

(4 marks)

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6 A differential equation is given by

$$(x^3 + 1)\frac{d^2y}{dx^2} - 3x^2\frac{dy}{dx} = 2 - 4x^3$$

(a) Show that the substitution

$$u = \frac{dy}{dx} - 2x$$

transforms this differential equation into

$$(x^3 + 1)\frac{du}{dx} = 3x^2u$$

(4 marks)

(b) Hence find the general solution of the differential equation

$$(x^3 + 1)\frac{d^2y}{dx^2} - 3x^2\frac{dy}{dx} = 2 - 4x^3$$

giving your answer in the form $y = f(x)$. *(8 marks)*

Turn over ▶

7 The curve C_1 is defined by $r = 2\sin\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$.

The curve C_2 is defined by $r = \tan\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$.

(a) Find a cartesian equation of C_1. *(3 marks)*

(b) (i) Prove that the curves C_1 and C_2 meet at the pole O and at one other point, P, in the given domain. State the polar coordinates of P. *(4 marks)*

(ii) The point A is the point on the curve C_1 at which $\theta = \frac{\pi}{4}$.

The point B is the point on the curve C_2 at which $\theta = \frac{\pi}{4}$.

Determine which of the points A or B is further away from the pole O, justifying your answer. *(2 marks)*

(iii) Show that the area of the region bounded by the arc OP of C_1 and the arc OP of C_2 is $a\pi + b\sqrt{3}$, where a and b are rational numbers. *(10 marks)*

END OF QUESTIONS